DEVELOPMENT OF A ROBUST DIFFUSION-KINEMATIC FLOW ALGORITHM FOR REGIONAL HYDROLOGIC MODELS OPERATING WITH LARGE TIME STEPS

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ABSTRACT

Diffusion flow models used to simulate hydrologic systems can become inaccurate under certain extreme flow conditions. Two such flow conditions include the relatively deep low slope flow condition as in the case of stagnant canals and the relatively shallow steep flow condition as in the case of mountain streams. In the case of relatively deep slow flows and short term simulations, the inertia effects are important, and full shallow water equations are needed to solve this problem. However, in the case of steep slopes, even if inertia terms can be neglected, long term simulations become problematic because of the need of fully implicit diffusion models to use small time steps when using kinematic flow. Many of the long term physically based hydrologic models for surface flow have to deal with meshes or cells when simulating irregular bottom slopes. These models need to trap water in deep ponds and pass water over steep slopes at the same time, creating conditions that are difficult to solve with models using only the diffusion flow approximation. Numerical methods that are stable with large time steps and small cells under both diffusion and kinematic flow conditions are needed in solving such problems.

The current paper describes a stable computational approach that can be useful in solving both kinematic and diffusion problems. A fully implicit finite volume formulation of the approximate St. Venant equations is used in the formulation. Newton's method is applied to solve the nonlinear equation resulting from this approach. The paper demonstrates the stability and the boundedness of the approach when using very large time steps of the order of days.

INTRODUCTION

The capabilities that are important for long term physically based surface water models using finite volume and finite element methods include the ability to (a) simulate both the kinematic and diffusion flows over steep and flat terrains using large time steps; (b) collect the correct amount of flow into basins defined by the cells or the mesh representing topography; (c) divert the correct volume of water into the streams that eventually flow into the ocean along the correct path, (d) collect the correct volume of water in ponds and wetlands created by the topography. Such models are not designed to focus on (i) the ability to provide detailed flow behavior and its variability within individual cells, (ii) the ability to simulate hydraulic jumps that occur when supercritical flow in steep slope meets subcritical flow in flat landscapes, (iii) the ability to simulate local flood peaks and flow impulses resulting from bores.

When diffusion and kinematic flow processes have to coexist in the same flow domain, the diffusion flow process is governed by a nonlinear parabolic partial differential equation (PDE) and the kinematic flow process is governed by a nonlinear hyperbolic PDE, both resulting from approximations of the St. Venant equations. Depending on the depth, the slope, and the magnitude of the physical parameters, these flows represent both supercritical and subcritical flows in the system. If the model is designed for multi-year simulations, large time steps are unavoidable if the run time is to be brought down, while smaller time steps are needed if the solution is to be stable.

Traditional diffusion flow models (Akan and Yen 1981, Lal 1998) use central differencing in the solution of the governing parabolic PDE's. However, central differencing methods are not suitable for solving hyperbolic problems. The result of using central difference methods to simulate kinematic flow can range from generation of nonlinear numerical oscillations to model instability. To avoid these problems, numerical methods for wet-bed and dry-bed overland flow have to solve a wide range of flow conditions from kinematic to diffusion while being extremely stable and robust with very large time steps.

Recent developments in the field of magneto hydro dynamics (MHD) and computational fluid dynamics (CFD) have contributed to the successful resolution of some of these challenges. There are new solution methods such as the TVD Lax-Friedrichs method (TVDLF) explained by Yee (1989) and Toth et al., (1998) that are useful in solving non-equilibrium type hyperbolic-parabolic problems using large time steps. Stable non-oscillatory fully implicit formulations are possible with these numerical schemes with which the final discrete form is solved using Newton's method. Both first order and second order spatial accuracies are possible with the approach. The TVD condition is used to make the solution essentially non-oscillatory. Some examples are shown in the paper.

GOVERNING EQUATIONS

St. Venant equations solved in the current models consist of an equation for conservation of mass and conservation of momentum. The equation for conservation of mass can be expressed in terms of the depth h and the discharge per unit width, q as

$$\frac{\partial h}{\partial t} + \frac{\partial q(h)}{\partial x} = 0 \tag{1}$$

Water level in this problem is expressed as $H = h + z$. The discharge q can be explained using the flow velocity u as $q = u(H - z)$. After neglecting the inertia terms, the equation for conservation of momentum reduces to $S_f = \frac{\partial H}{\partial x}$ where $S_f =$ friction slope defined using the Manning equation. The Manning equation is used to derive an expression for q in terms of h and S_f.

$$q = \frac{h^{\frac{5}{3}}\sqrt{S_f}}{n_b} \tag{2}$$

where n_b = Manning constant; $S_f = \frac{\partial H}{\partial x}$ for diffusion flow; $S_f = \frac{\partial z}{\partial x}$ for kinematic flow. Linearization of (1) and (2) result in the following expressions that is useful in the analysis.

$$c = \frac{\partial q}{\partial h} = \frac{5}{3}\frac{h^{\frac{2}{3}}\sqrt{S_0}}{n_b} \tag{3}$$

$$K_d = \frac{\partial q}{\partial S_0} = \frac{1}{2}\frac{h^{\frac{5}{3}}}{n_b\sqrt{S_0}} \tag{4}$$

where c = celerity of the wave; K_d = hydraulic diffusivity.

THE NUMERICAL FORMULATIONS

The scalar conservative equation (1) without the source term is solved using a finite volume formulation that is derived in terms of flux functions as

$$\frac{\partial h_i}{\partial t} = \frac{\Delta t}{\Delta x}(q^{n+1}_{i-\frac{1}{2}} - q^{n+1}_{i+\frac{1}{2}}) \tag{5}$$

where $q^{n+1}_{i+\frac{1}{2}}$ = numerical flux function at wall $i+\frac{1}{2}$. Figure 1 shows a definition sketch. The flux function has to follow the rule $q(h_i, h_i) = q(h_i)$ to enforce the consistence condition for the scheme. Both the diffusion flow algorithm and the TVDLF algorithm described below are based on this equation.

The diffusion flow model

When using the diffusion flow method, the diffusion flux is described using

$$q_{i+\frac{1}{2}} = -\frac{1}{2}\frac{h^{\frac{5}{3}}_{i+\frac{1}{2}}}{\max(|S_{i+\frac{1}{2}}|^{\frac{1}{2}}, \delta_s)}\operatorname{sgn}(S_{i+\frac{1}{2}}) \tag{6}$$

where $h_{i+\frac{1}{2}}$ is defined as $max(0, 0.5(h_i + h_{i+1}))$; δ_s is used to avoid the singularity; $S_{i+\frac{1}{2}}$ is defined as

$$S_{i+\frac{1}{2}} = \frac{h_{i+1} + z_{i+1} - h_i - z_i}{\Delta x} \tag{7}$$

A value of $\delta_s = 10^{-5}$ is suitable for many applications. There are many ways to calculate $h_{i+\frac{1}{2}}$ that are not discussed here.

The Total Variation Diminishing Lax-Friedrichs Scheme (TVDLF)

The purpose of the TVD formulation developed by Harten (1983) is to prevent the generation of spurious oscillations. Yee (1989) explained a number of design principles for TVD schemes. TVD property automatically preserves monotonicity and prevents creation of spurious oscillations. The condition for a numerical solution to be TVD is

$$TV(h^{n+1}) \leq TV(h^n) \tag{8}$$

where the total variation $TV(h)$ is defined as

$$TV(h^n) = \sum_{j=-\infty}^{\infty} |h_{j+1}^n - h_j^n| = \sum_{j=-\infty}^{\infty} |\Delta h_{j+\frac{1}{2}}^n| \tag{9}$$

and $\Delta h_{j+\frac{1}{2}}^n = h_{j+1}^n - h_j^n$. It is possible to obtain a first order TVD scheme by modifying the central difference scheme to be an upwind scheme or by limiting the flux. It has been shown that the Lax-Friedrichs method satisfies the TVD property.

The numerical flux function $q_{i+\frac{1}{2}}$ in (1) for the TVDLF method can be calculated using a variety of methods. The method used here is

$$q_{i+\frac{1}{2}} = \frac{1}{2}(q_i + q_{i+1}) - \phi_{i+\frac{1}{2}} \tag{10}$$

For surface water flow, this can be written as

$$q_{i+\frac{1}{2}} = -\frac{1}{2}\left[h_i^{\frac{5}{3}} + h_{i+1}^{\frac{5}{3}}\right]\frac{|S_{i+\frac{1}{2}}|^{\frac{1}{2}}}{n_b}\mathrm{sgn}(S_{i+\frac{1}{2}}) - \frac{1}{\Delta x}|c_{i+\frac{1}{2}}^{max}|(h_{i+1} - h_i) \tag{11}$$

In the 1-D case, a tri-diagonal system of equations can be obtained from this using (11) when the Newton's method is applied to solve the finite volume approximation of the governing equation (1). Thomas algorithm is useful in solving the linear equation resulting from this approach. The tri-diagonal system can be defined as

$$\begin{aligned}
a_i &= +\frac{\partial q_{i-\frac{1}{2}}}{\partial h_{i-1}} && (12)\\
b_i &= 1 + \frac{\partial q_{i-\frac{1}{2}}}{\partial h_i} - \frac{\partial q_{i+\frac{1}{2}}}{\partial h_i} && (13)\\
c_i &= -\frac{\partial q_{i+\frac{1}{2}}}{\partial h_{i+1}} && (14)\\
r_i &= q_{i-\frac{1}{1}} - q_{i+\frac{1}{2}} && (15)
\end{aligned}$$

The right hand side r_i can be expanded as

$$r_i = \frac{\Delta t}{2}\left[q_{i-1} + q_i - \frac{1}{\Delta x}|c_{i-\frac{1}{2}}|^{max}(h_i - h_{i-1})\right] - \frac{\Delta t}{2}\left[q_i + q_{i+1} - \frac{1}{\Delta x}|c_{i+\frac{1}{2}}|^{max}(h_{i+1} - h_i)\right] \tag{16}$$

An analytical estimate can be obtained for the Jacobian for the specific case of (10) or (11) as

$$\begin{aligned}
a_i &= -\frac{1}{2}\frac{\Delta t}{\Delta x}(c_{i-\frac{1}{2}} + |c_{i-\frac{1}{2}}|^{max} + \frac{1}{\Delta x}K_{i-\frac{1}{2}}) && (17)\\
b_i &= 1 - \frac{1}{2}\frac{\Delta t}{\Delta x}(|c_{i-\frac{1}{2}}|^{max} + |c_{i+\frac{1}{2}}|^{max} + \frac{1}{\Delta x}(K_{i-\frac{1}{2}} + K_{i+\frac{1}{2}})) && (18)\\
c_i &= -\frac{1}{2}\frac{\Delta t}{\Delta x}(-c_{i+\frac{1}{2}} + |c_{i+\frac{1}{2}}|^{max} + \frac{1}{\Delta x}K_{i+\frac{1}{2}}) && (19)
\end{aligned}$$

and the system of linear equations is represented as

$$a_i x_{i-1} + b_i x_i + c_i x_{i+1} = r_i \tag{20}$$

and r_i = right hand side vector component.

NUMERICAL EXPERIMENTS

Numerical experiments are carried out to demonstrate certain behaviors of the diffusion flow method and the stability of the TVDLF method under extreme numerical conditions and with large time steps. One dimensional tests are used in both cases. To illustrate the numerical oscillations in diffusion flow models when simulating flow in steep canals, an experiment is conducted using a 1-D model with 500 m long and 870 m wide segments. A discharge rate of 10 m^2/s is applied at the upstream end, and a uniform flow condition is allowed at the downstream end. When a time steps of one day is used, this model is stable only when the Manning constant exceeds an unrealistic value of about 70. Figure 2 shows the solution when the Manning roughness is 70. In the figure, the water depth fluctuates between 4.9 m and 2.3 m in the middle of the canal while the analytical solution for steady state depth is 3.78 m. The two snap shots shown in Figure 2 are 6 days apart. An animation of the solution shows that perfectly normal-looking water waves of a 12 day period pass down the canal during the simulation. Experimentation with variable time steps shows that these oscillations are created due to numerical issues. Nonlinear waves in numerical models are explained by Yee and Sewby (1992). The results show that diffusion flow models can be subjected to numerical problems in addition to the applicability problems specified by Ponce et al., (1978) and truncation error problems described by Lal (2000).

Figure 3 shows the results of a test carried out using the TVDLF method. The 1-D model used for the test has segments of length 100 m and arbitrary bottom elevations. The Manning roughness used in the experiment is 0.03. Results show that the TVDLF method is stable with one day time steps even if the inertia terms are neglected in the formulation. The results also show that ponding takes place in the correct location and steep flows are simulated without oscillations. These are desired behaviors of any numerical method used for the type of wet and dry bottom conditions present in surface water models. The experimentation has to continue to show that the method is accurate for a wide range of conditions that are valuable to hydrologists.

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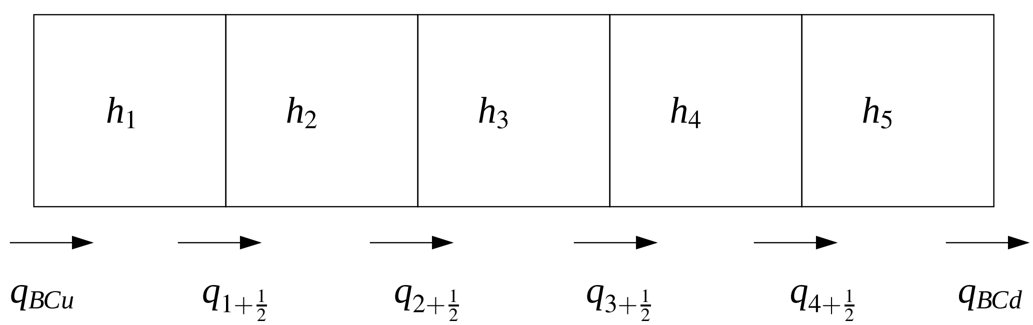

Figure 1: Mesh cells in a five cell 1-D problem

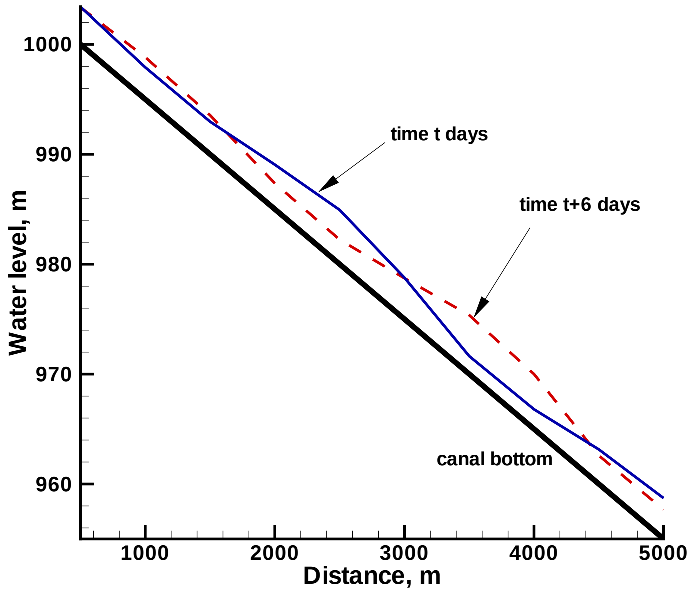

Figure 2: Numerical waves in 1-D uniform flow. Slope=0.01, segment length=50.0 m, time step=1 day, $n_b = 70$

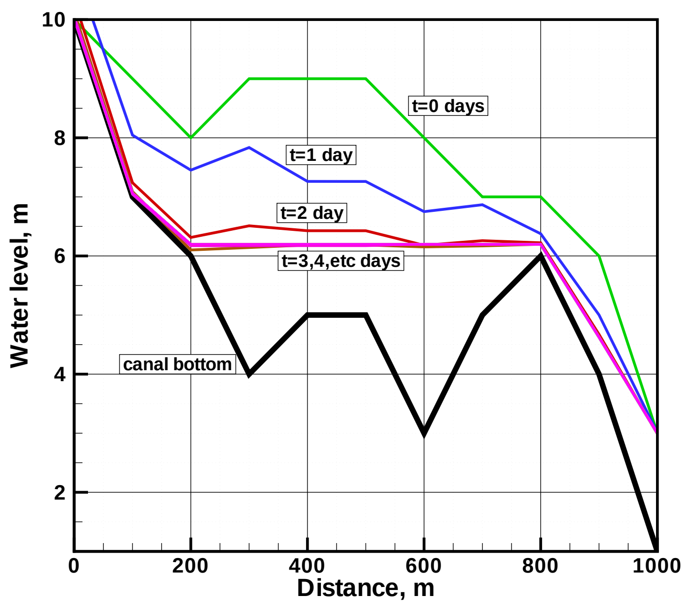

Figure 3: Application of the TVDLF method for a 1-D problem with an irregular bottom using a time step of 1 day